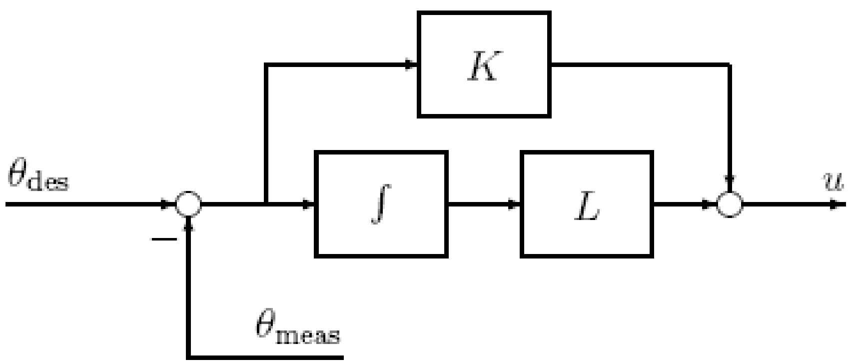
Departamentul de MECATRONICĂ
Facultatea de MECANICĂ
UNIVERSITATEA POLITHENICA TIMIŞOARA

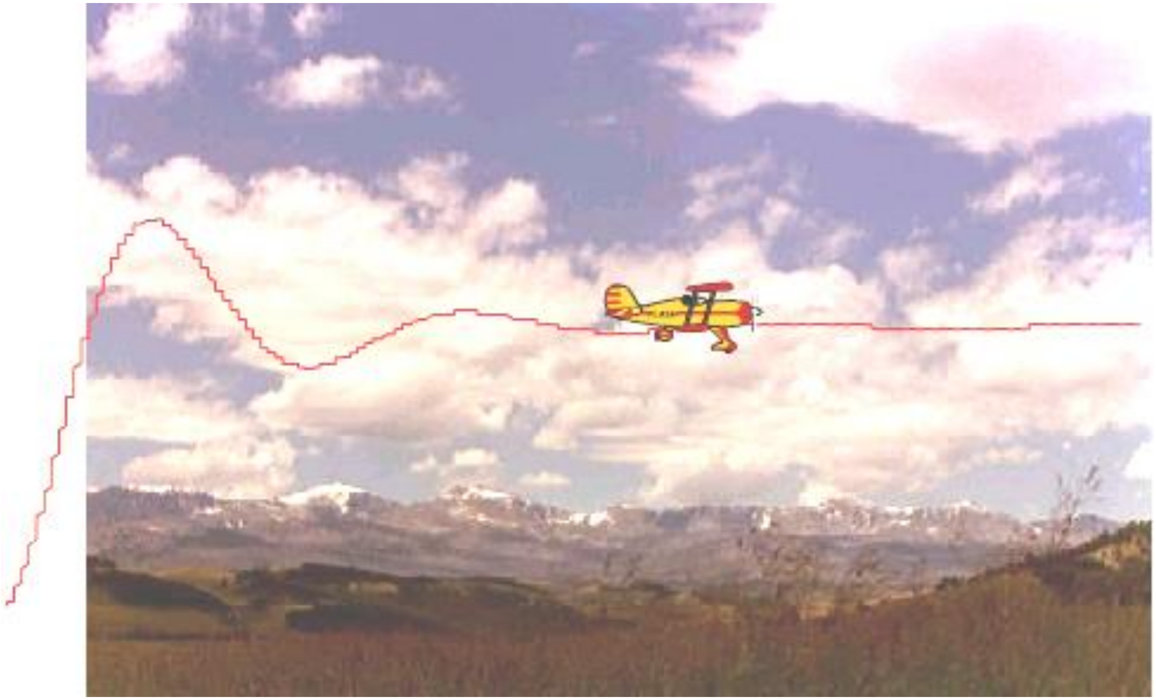
TEORIA SISTEMELOR AUTOMATE



Prof. dr. ing. Valer DOLGA,

Cuprins_8
Analiza si raspunsul sistemelor liniare in domeniul timp II

Sisteme de ordinul 2



Exemplu de sistem mecatronic cu o comportare tipică de sistem de ordinul doi

Ecuaţia diferenţială care descrie sistemul de ordinul doi:

$$a_2 \cdot \frac{d^2 y}{dt^2} + a_1 \cdot \frac{dy}{dt} + a_0 \cdot y = b_0 \cdot u$$

$$G(s)=\frac{Y(s)}{U(s)}=\frac{b_0}{a_2\cdot s^2+a_1\cdot s+a_0}$$

$$G(s)=\frac{Y(s)}{U(s)}=\frac{b_0}{a_2\cdot s^2+a_1\cdot s+a_0}=\frac{\frac{b_0}{a_0}}{\frac{a_2}{a_0}\cdot s^2+\frac{a_1}{a_0}\cdot s+1}=\frac{\frac{b_0}{a_0}}{\frac{1}{\frac{a_0}{a_2}}\cdot s^2+2\frac{a_1}{2\cdot\sqrt{a_0\cdot a_2\cdot\frac{a_0}{a_2}}}\cdot s+1}=$$

$$=\frac{S}{\frac{1}{\omega_n^2}\cdot s^2+\frac{2\xi}{\omega_n}\cdot s+1}=\frac{S\cdot\omega_n^2}{s^2+2\xi\omega_n\cdot s+\omega_n^2}$$

$S=\frac{b_0}{a_0}$ **- sensibilitatea sistemului,** sau **factorul de amplificare**

$\xi=\frac{a_1}{2\sqrt{a_0\cdot a_2}}$ **- factorul de amortizare**

$\omega_n^2=\frac{a_0}{a_2}$ **- pulsaţia naturală a sistemului**

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Sisteme de ordinul 2 - raspunsul la semnal impuls unitar

$$G(s) = \frac{S}{\frac{1}{\omega_n^2} \cdot s^2 + \frac{2\xi}{\omega_n} \cdot s + 1} = \frac{S}{T^2 \cdot s^2 + 2\xi T \cdot s + 1}$$

$$T = 1/\omega_n$$

Raspunsul sistemului la un semnal standard

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{b_0}{a_2 \cdot s^2 + a_1 \cdot s + a_0} \cdot U(s)\right\}$$

❖**Semnal – impuls unitar**

U(s)=1

$$y(t) = \mathcal{L}^{-1}\left\{\frac{S\omega_n^2}{s^2 + 2\xi\omega_n \cdot s + \omega_n^2} \cdot 1\right\}$$

a) **cazul a două rădăcini reale distincte**

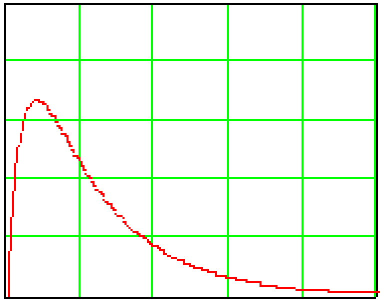
$$\Delta = b^2 - 4ac = 4\omega_n^2 \cdot \left(\xi^2 - 1\right) > 0 \quad \Rightarrow \quad \xi > 1$$

$$y(t) = S\omega_n^2 \cdot \mathcal{L}^{-1}\left\{\frac{1}{(s - s_1)(s - s_2)}\right\}$$

$$s_1 = \omega_n(-\xi + \sqrt{\xi^2 - 1}), \quad s_2 = \omega_n(-\xi - \sqrt{\xi^2 - 1})$$

$$y(t) = S\omega_n^2\left(\frac{1}{s_1 - s_2}e^{s_1 t} - \frac{1}{s_1 - s_2}e^{s_2 t}\right)$$

Raspunsul calitativ al sistemului →



- t=0, răspunsul este nul.
- există un maxim (şi numai unul), iar apoi răspunsul tinde asimptotic la 0;
- valoarea de maxim se determină din condiţia anulării primei derivate:

$$\frac{dy(t)}{dt} = \omega_n^2\left(\frac{s_1}{s_1 - s_2}e^{s_1 t} - \frac{s_2}{s_1 - s_2}e^{s_2 t}\right) = 0$$

$$s_1 e^{s_1 t_{vf}} = s_2 e^{s_2 t_{vf}}$$

$$t_{vf} = -\ln\frac{s_1 / s_2}{s_1 - s_2}$$

Un astfel de sistem poate fi înlocuit cu o serie de două sisteme de ordinul 1.

b) **cazul a două rădăcini complexe** (factor de amortizare) $\xi < 1$

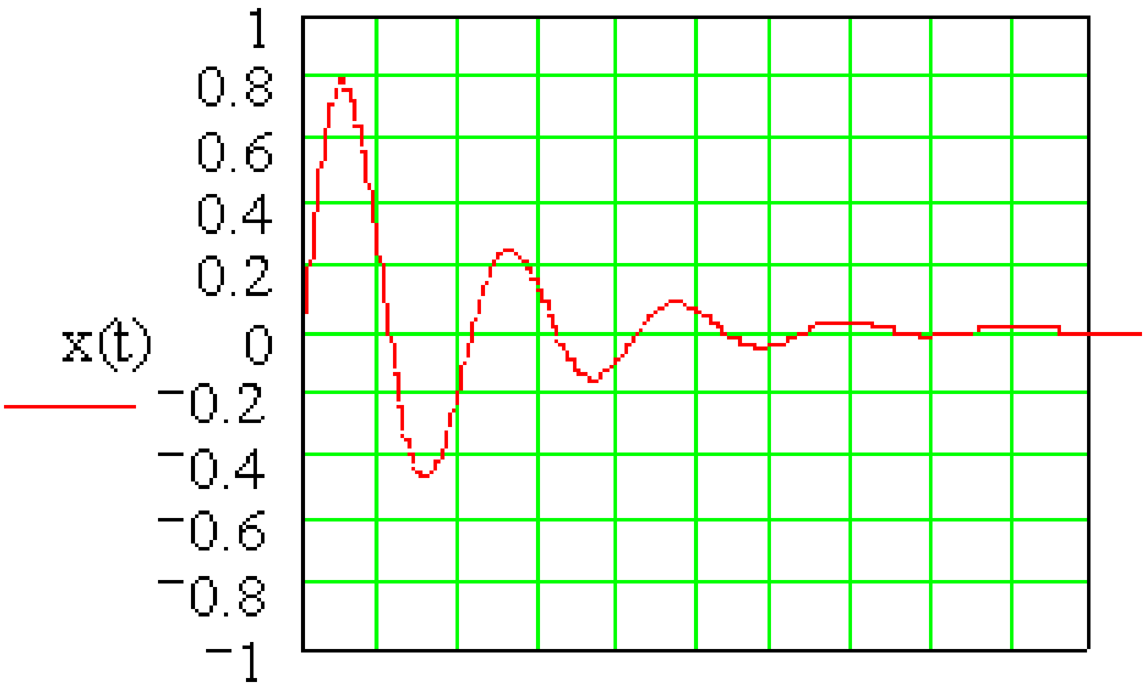
Raspunsul sistemului:

$$y(t) = Ae^{-\xi\omega_n t}\,\boldsymbol{sin(}\,\omega_n\sqrt{1-\xi^2}\,t+\phi\boldsymbol{)}$$

- constantele "A" şi "φ" se determină din condiţiile iniţiale:
- la momentul t=0 semnalul de ieşire trebuie să fie nul, y(t)=0, defazajul φ rezultă nul;
- "A" se obţine considerând şi derivata semnalului de ieşire în origine.

$$A = \frac{S\omega_n}{\sqrt{1-\xi^2}}$$

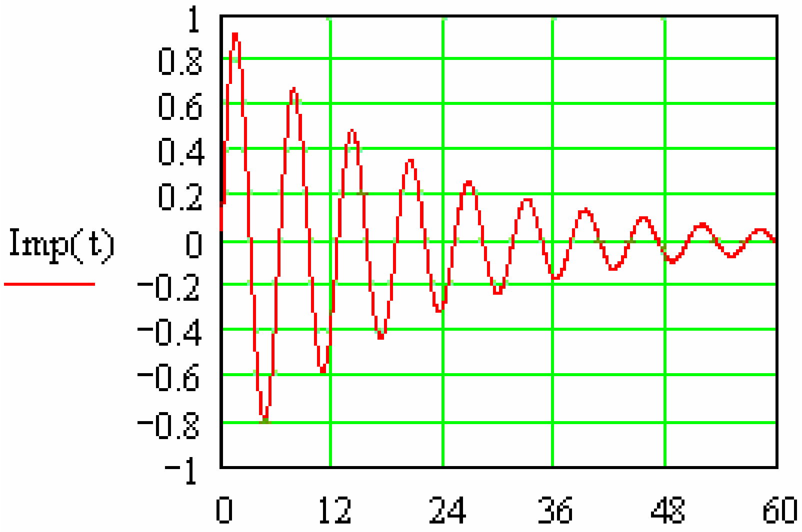
$$\Rightarrow \quad y(t) = \frac{S\omega_n}{\sqrt{1-\xi^2}}\,e^{-\xi\omega_n t}\cdot \boldsymbol{sin(}\,\omega_n(\sqrt{1-\xi^2})t\boldsymbol{)}$$



Răspunsul sistemului de ordinul doi la impuls unitar pentru cazul când factorul de amortizare este subunitar $\xi < 1$

- În ambele cazuri, pulsatia oscilaţiei mărimii de ieşire este totdeauna mai mică decât cea naturală ω_n:

$$\omega_n \sqrt{\left|1-\xi^2\right|}$$



a) ξ=0.05

b) ξ=0.95

Răspunsul oscilatoriu al unui sistem de ordinul doi la impuls unitar, pentru diferiţi factori de amortizare

Sisteme de ordinul 2 – raspunsul la semnal treapta

Se preferă utilizarea funcţiei de transfer în forma:

$$Y(s) = G(s)X(s)$$

$$X(s) = \frac{1}{s}$$ - semnal treapta

$$Y(s) = \frac{k\omega_n^2}{s\left(s^2 + 2\xi\omega_n s + \omega_n^2\right)}$$

$$Y(s) = \frac{k\omega_n^2}{s(s + p_1)(s + p_2)}$$

$$s^2 + 2\xi\omega_n + \omega_n^2 = 0$$

$$p_{1,2} = \frac{-2\xi\omega_n \pm \sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2}$$

$$\begin{cases} p_1 = -\xi\omega_n + \sqrt{\xi^2\omega_n^2 - \omega_n^2} = -\xi\omega_n + \omega_n\sqrt{\xi^2 - 1} \\ p_2 = -\xi\omega_n - \sqrt{\xi^2\omega_n^2 - \omega_n^2} = -\xi\omega_n - \omega_n\sqrt{\xi^2 - 1} \end{cases}$$

$$\xi > 1$$

$$y(t) = \frac{k\omega_n^2}{p_1 p_2}\left(1 - \frac{p_2}{p_2 - p_1}\cdot e^{-p_1 t} + \frac{p_1}{p_2 - p_1}\cdot e^{-p_2 t}\right)$$

$$t \to \infty \qquad y(t) = \frac{k\omega_n^2}{p_1 p_2}$$

$$\xi = 1$$

$$p_1 = p_2 = -\omega_n$$

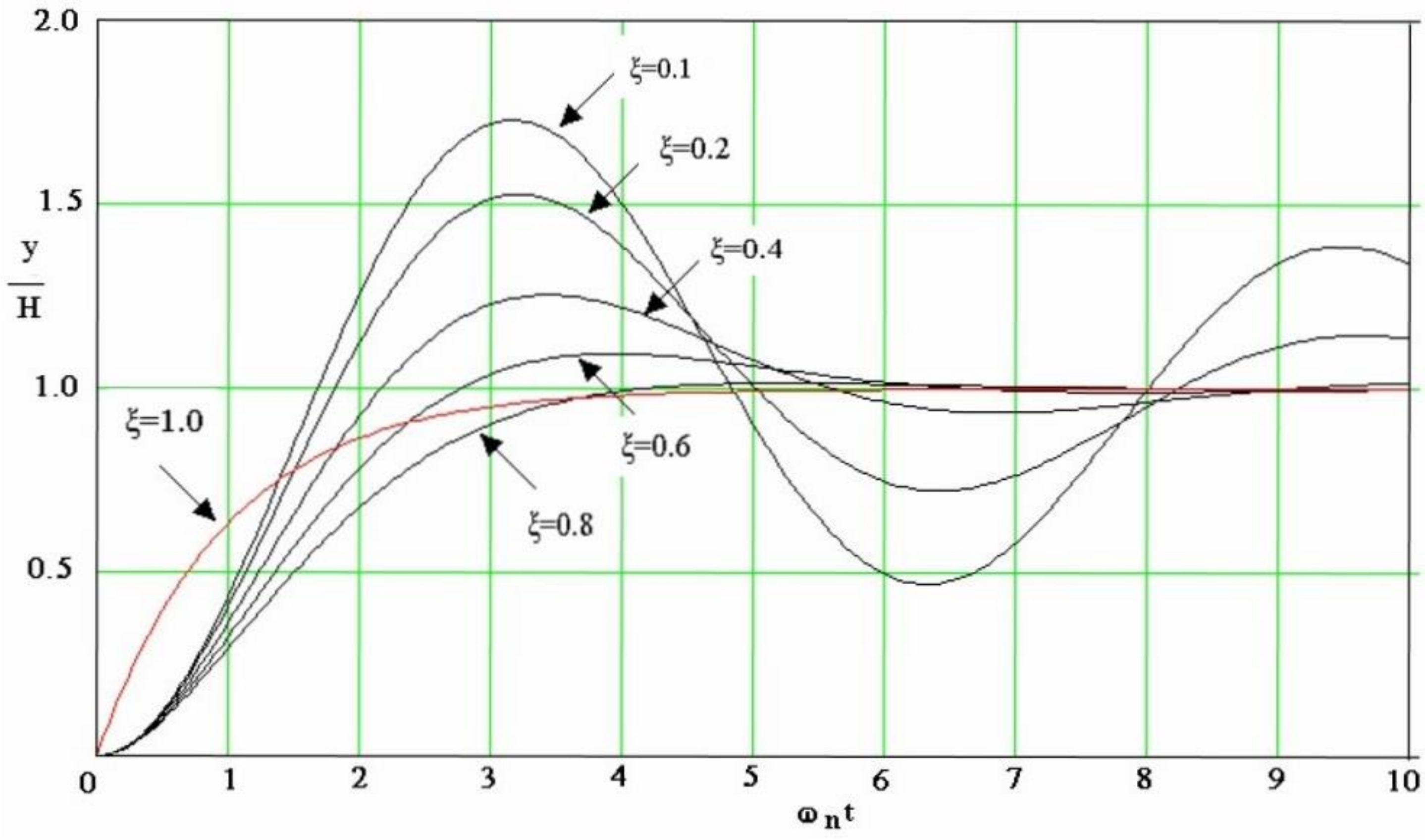
$$L^{-1}\left\{\frac{1}{(s+a)^2}\right\} = t\cdot e^{-at}$$

$$Y(s) = \frac{k\omega_n^2}{s(s+\omega_n)^2}$$

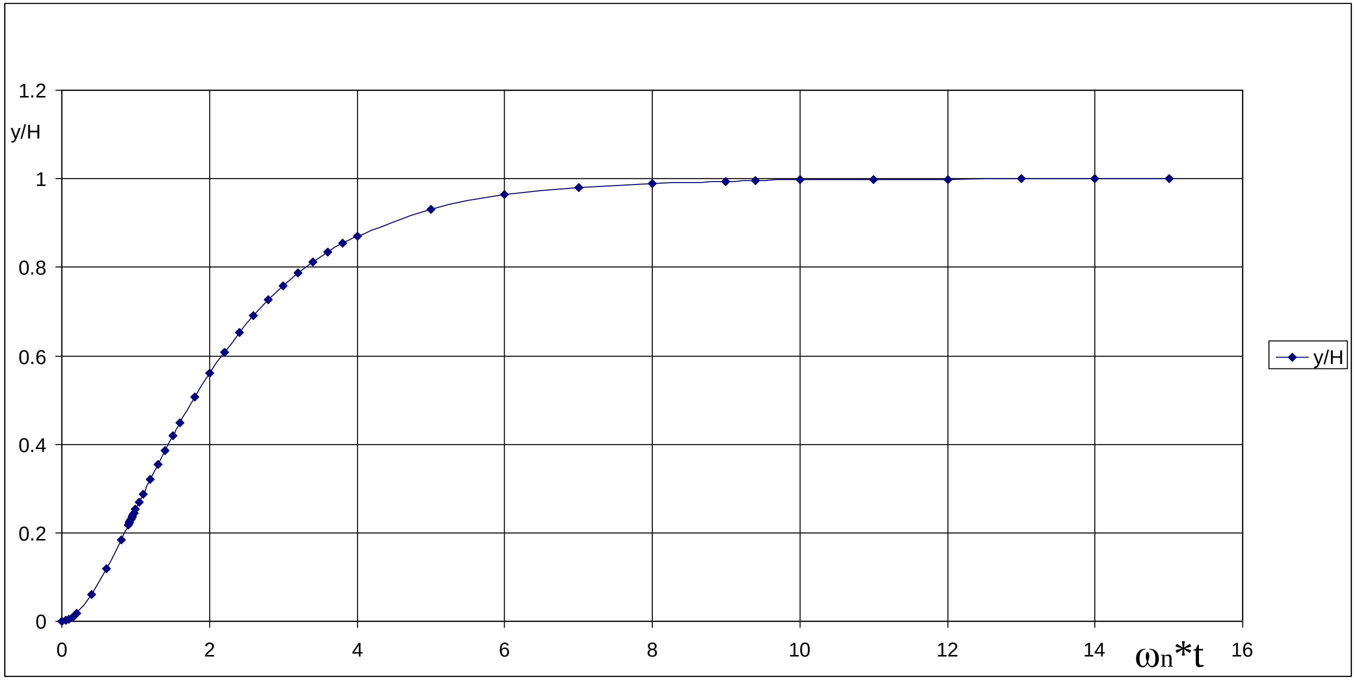
$$Y(s) = k\left(\frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2}\right) \Rightarrow y(t) = k\left(1 - e^{-\omega_n t} - \omega_n t\cdot e^{-\omega_n t}\right)$$

$$\xi < 1$$

$$y(t) = k\left[1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin\left(\omega_n\sqrt{1-\xi^2}\,t + \phi\right)\right] \qquad \cos\phi = \xi$$



- pentru cazul $\xi = 1$, se obţine amortizarea critică.
- pentru $\xi > 1$, aspectul curbei este cel din figura



Răspunsul indicial al unui sistem de ordin 2 cu factor de amortizare $\xi = 1.1$

Sisteme de ordinul 2 – raspunsul la semnal rampa

Pentru o mărime de intrare de tip:

$$u(t) = ct$$

⬇

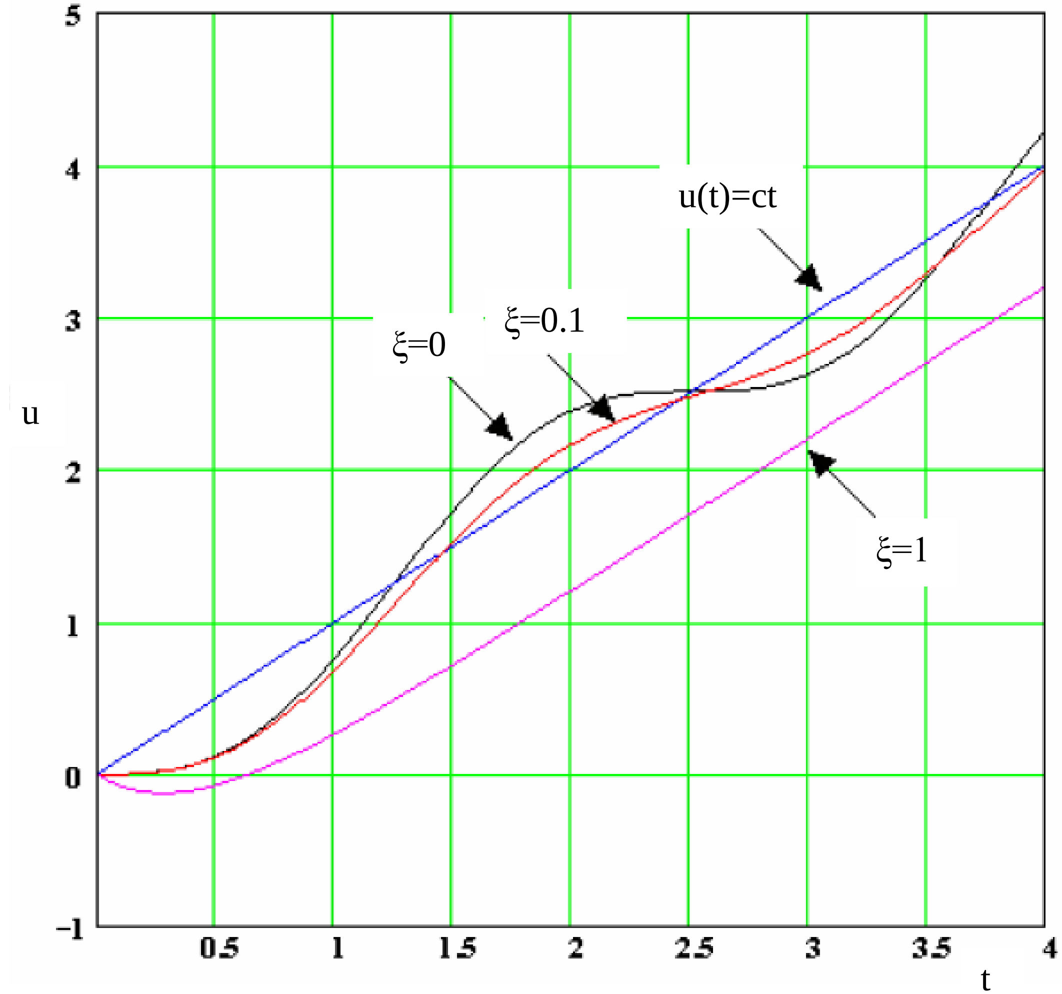
$$U(s) = c / s^2$$

Se utilizeaza forma standard pentru functia de transfer:

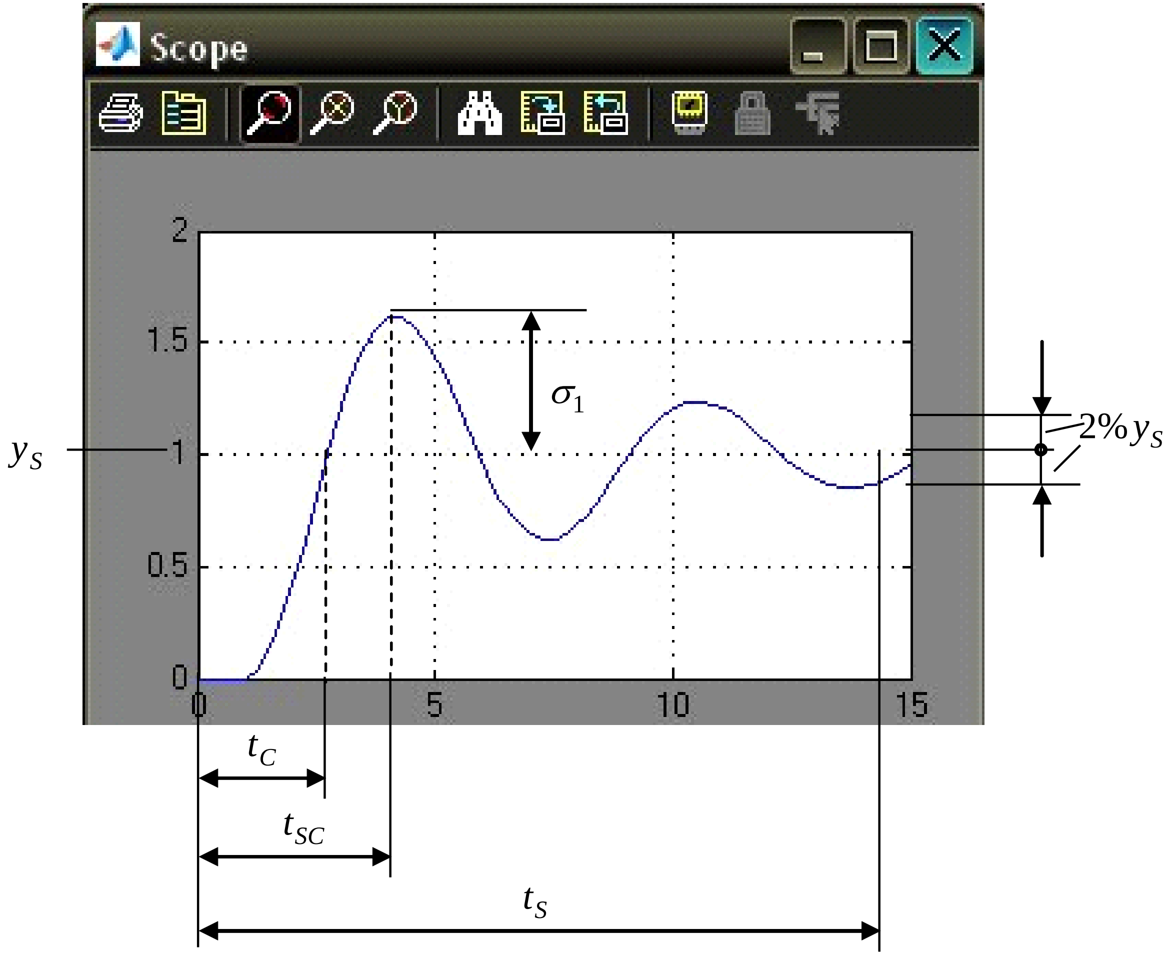
$$G(s) = \frac{1}{T^2 s^2 + 2\xi T s + 1}$$

⬇

$$Y(s) = \frac{c}{s^2} \cdot \frac{1}{T^2 s^2 + 2\xi T s + 1}$$



Răspunsul unui sistem de ordinul 2 la un semnal rampă



(y_S – valoarea de regim stabilizat; σ – supracreşterea; t_C – timpul de creştere; t_{SC} – timpul de supracreştere; t_S – timpul de stabilizare).

$$t_C = \frac{\pi}{2\omega_0\sqrt{1-\xi^2}}$$

Timpul de crestere – durata intervalului de crestere de la 10 % la 90 % din valoarea de regim stabilizat

$$t_{SC} = \frac{\pi}{\omega_0\sqrt{1-\xi^2}}$$

Timpul de atingere a valorii maxime

$$\sigma_1 = y_S \cdot e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}}$$

Supracresterea

$$rap\_amortizare = \frac{\sigma_2}{\sigma_1} = \frac{y_S \cdot e^{-\frac{3\pi\xi}{\sqrt{1-\xi^2}}}}{y_S \cdot e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}}} = e^{-\frac{2\pi\xi}{\sqrt{1-\xi^2}}}$$

$$t_S \approx \frac{4}{\xi\omega_0}$$

Durata de linistire (eroarea admisa pentru atingerea valorii stabilizate - 2 %)

$$t_S = \frac{3}{\xi\omega_n}$$

Durata de linistire (eroarea admisa pentru atingerea valorii stabilizate - 5 %)

$$n = \frac{2}{\pi}\sqrt{\frac{1}{\xi^2} - 1}$$

Numarul de oscilatii pina la linistire

Probleme propuse:

Descrieti forma semnalului de iesire a unui element senzorial cu factorul de amortizare egal cu:

a) 0;

b) 0.5;

c) 1;

d) 1.5.

Un element senzorial are frecventa de rezonanta 100 Hz si coeficientul de amortizare egal cu 0.6. Se cere sa se determine:

- supracresterea [%];
- timpul de crestere la o variatie brusca a semanlului de intrare;
- raportul de amortizare;
- durata de linistire;
- numarul de oscilatii pina la linistire;

Exemple

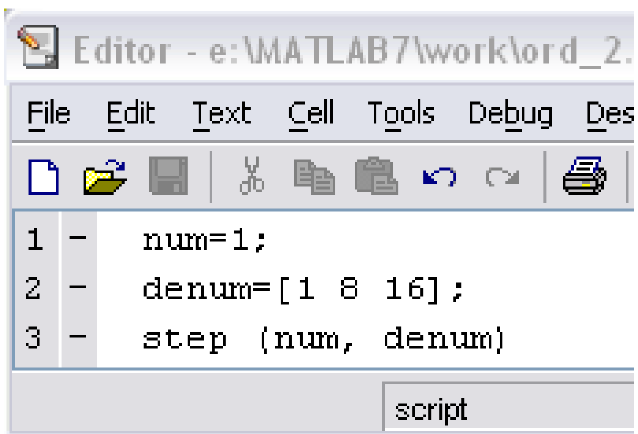
$$G(s)=\frac{1}{s^2+8s+16}$$

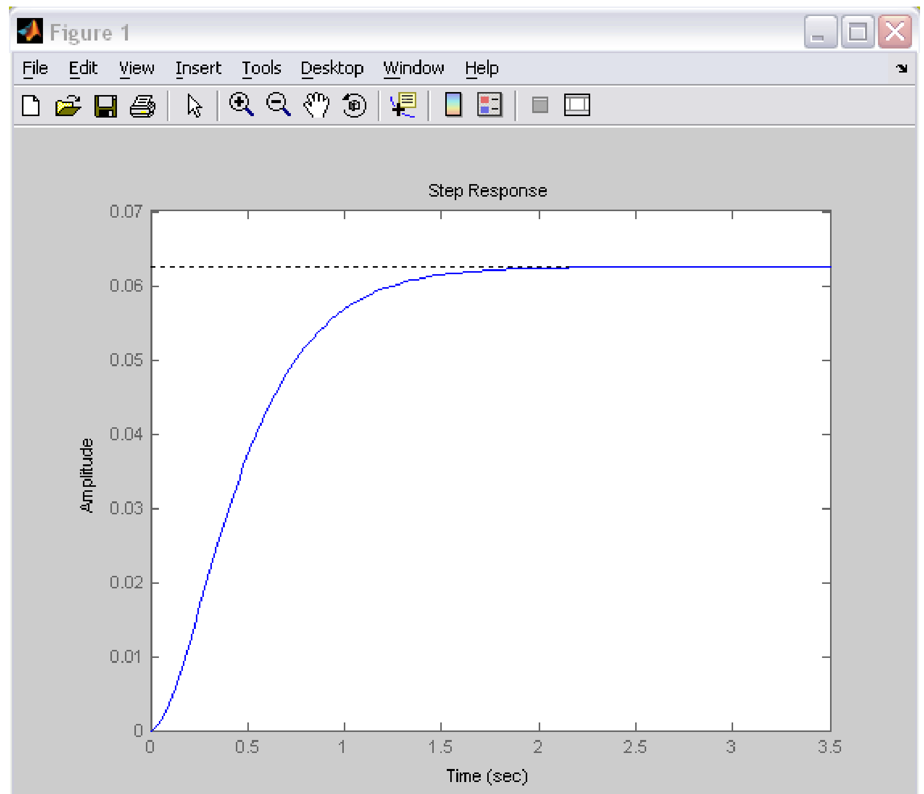
$$Y(s)=G(s)X(s)$$

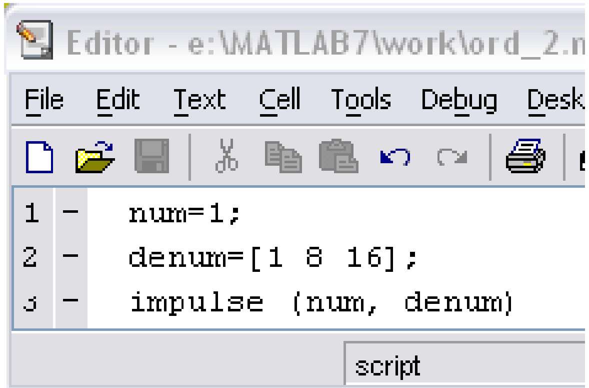
$$X(s)=\frac{1}{s}$$

$$p_1=p_2=-4$$

$$Y(s)=\frac{1}{s\left(s^2+8s+16\right)}=\frac{1}{s(s+4)(s+4)}$$






Editor - e:\MATLAB7\work\ord_2.m
File Edit Text Cell Tools Debug Desk
1 – num=1;
2 – denum=[1 8 16];
3 – impulse (num, denum)
script

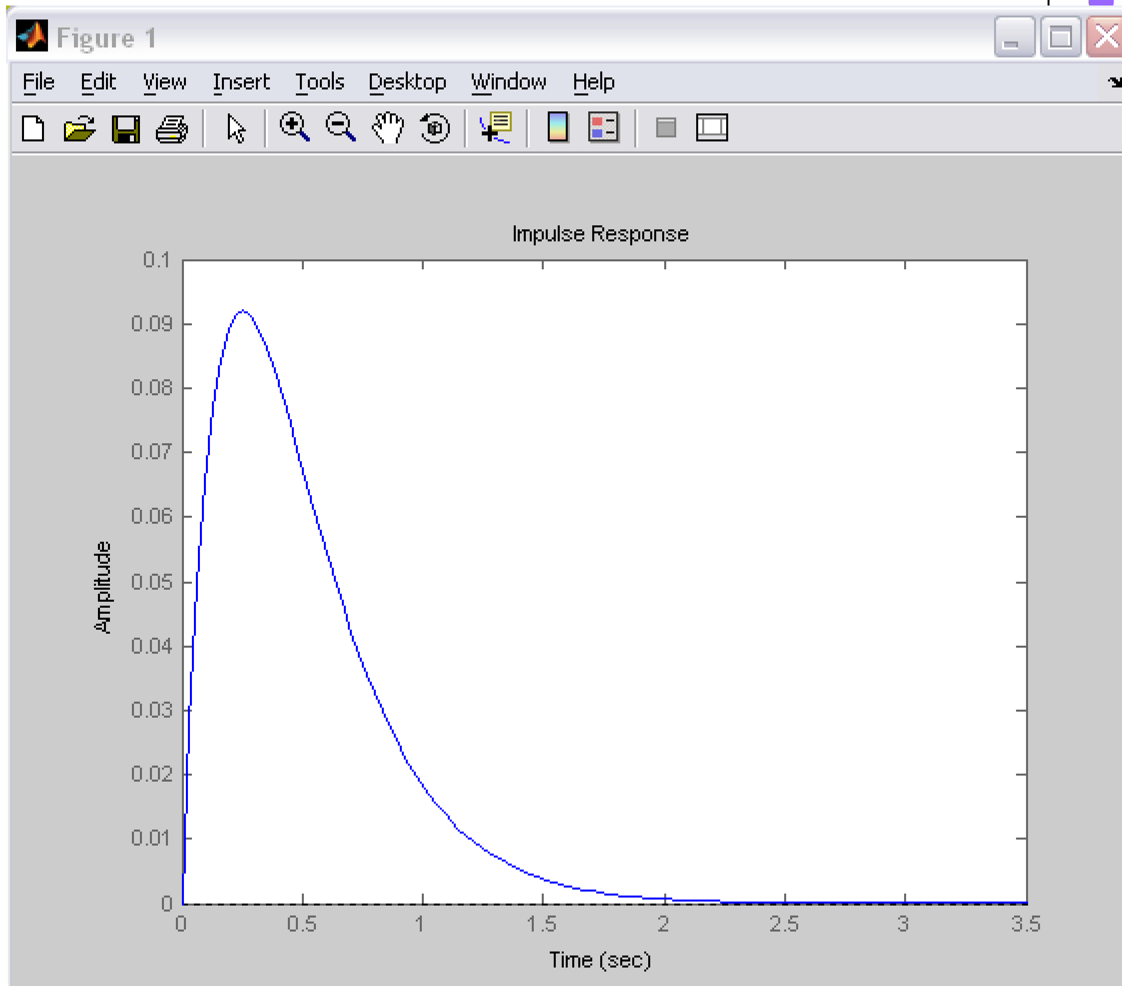


Figure 1
File Edit View Insert Tools Desktop Window Help
Impulse Response
Amplitude
Time (sec)
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.1
0 0.5 1 1.5 2 2.5 3 3.5

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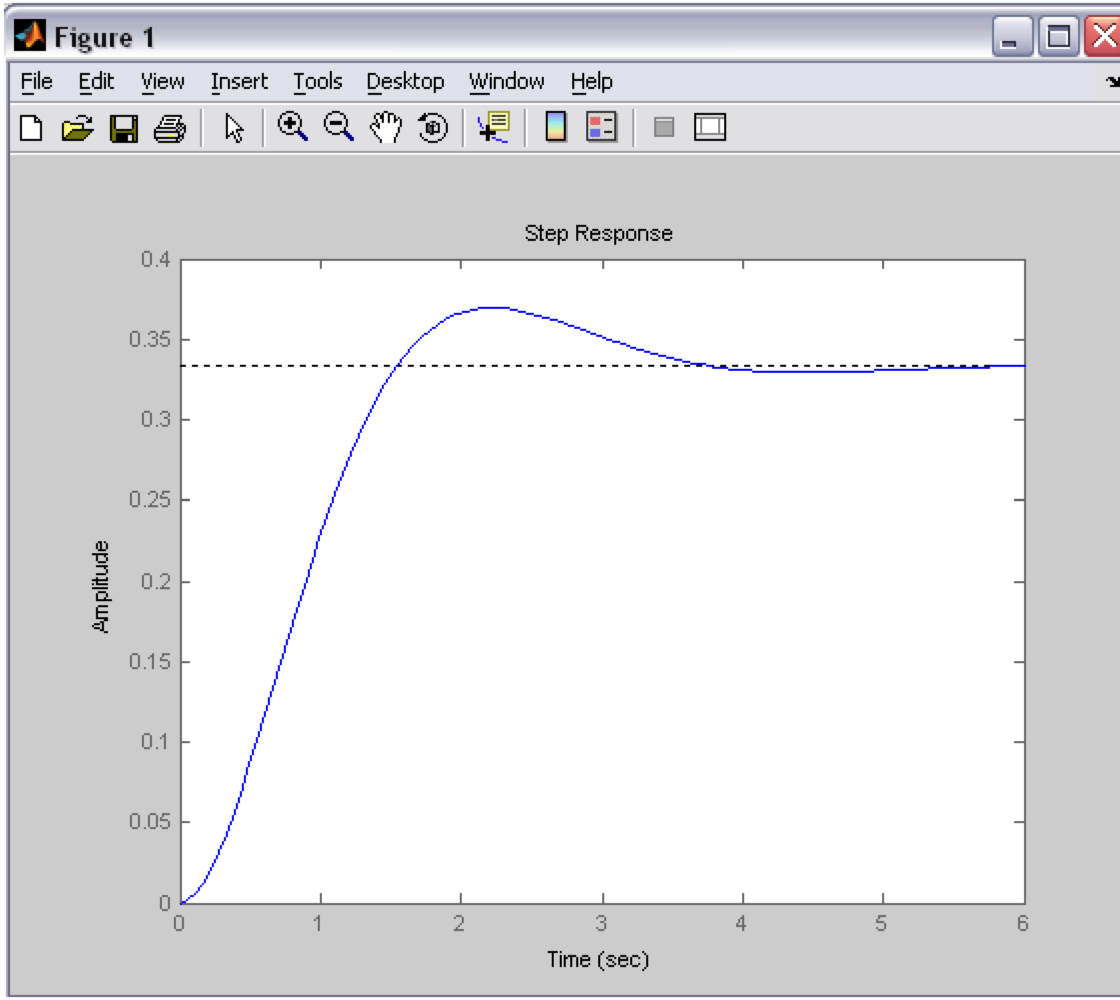
```
num=1;
denum=[1 2 3];
roots (denum)
step(num, denum)
```

```
ans =

  -1.0000 + 1.4142i
  -1.0000 - 1.4142i
```



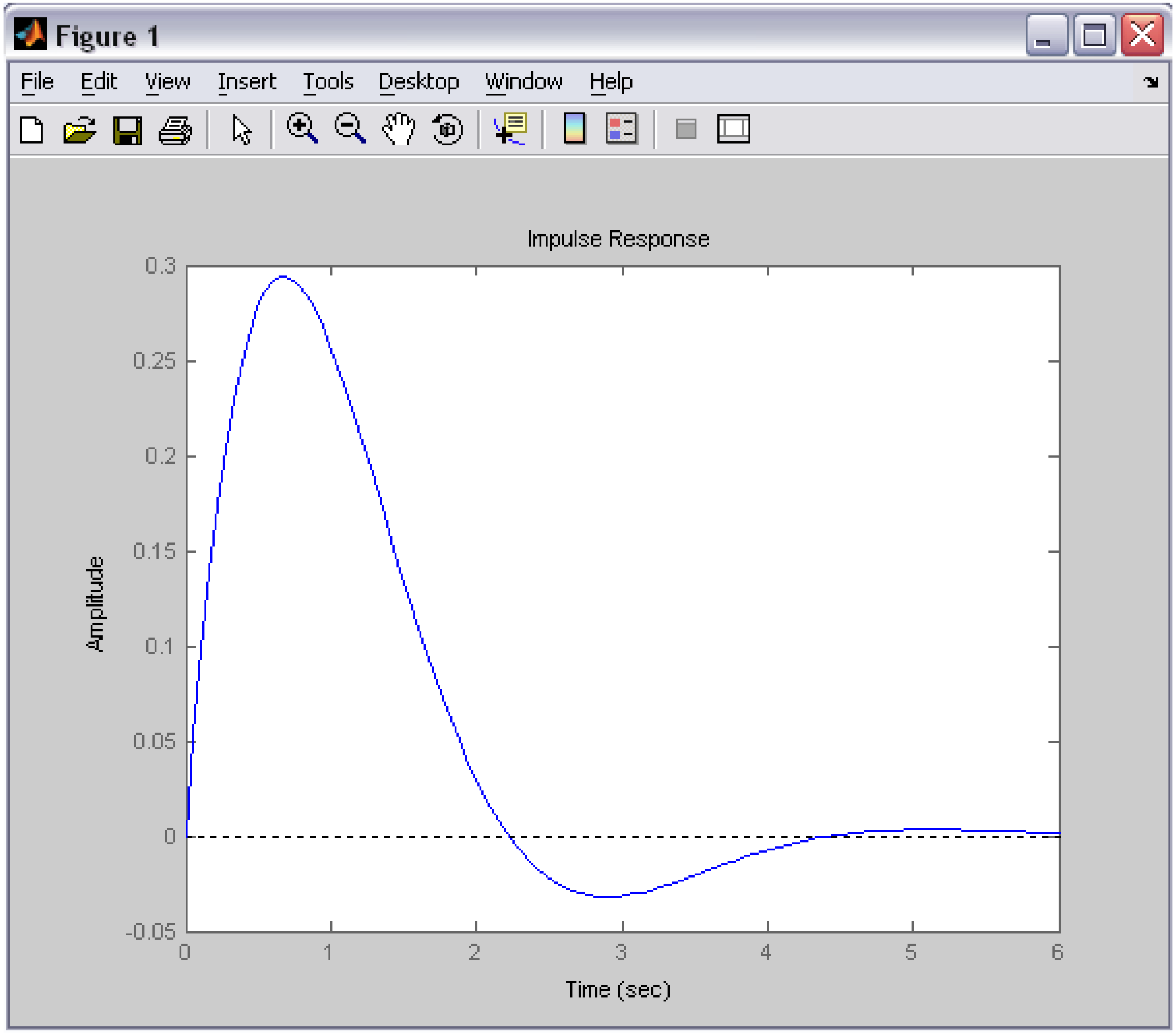


Figure 1
File
Edit
View
Insert
Tools
Desktop
Window
Help
Impulse Response
Amplitude
Time (sec)
0.3
0.25
0.2
0.15
0.1
0.05
0
-0.05
0
1
2
3
4
5
6

Exemplu_2

Să se determine răspunsul la semnal treaptă a unui sistem de ordinul 2 pentru factor de amortizare nul ($\xi=0$).

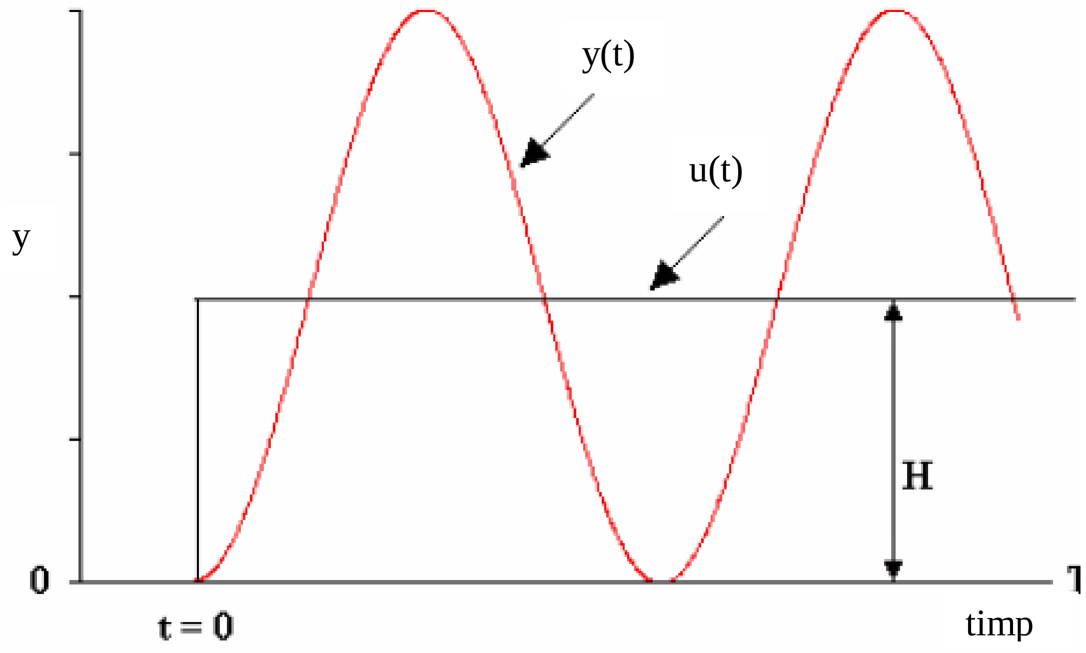
- Pentru $\xi=0$, coeficientul σ rezultă de asemenea nul ($\sigma=\xi\omega_n=0$).

$$\Rightarrow \quad G(s)=\frac{\omega_n^2}{s^2+\omega_n^2} \quad \Rightarrow \quad Y(s)=G(s)\cdot\frac{H}{s}=\frac{H\omega_n^2}{s(s^2+\omega_n^2)}$$

$$\mathcal{L}\left[\int_0^t x(\tau)d\tau\right]=\frac{F(s)}{s}$$

$$\mathcal{L}^{-1}\left(\frac{H}{s}\frac{\omega_n^2}{s^2+\omega_n^2}\right)=H\omega_n\mathcal{L}^{-1}\left(\frac{\frac{\omega_n}{s^2+\omega_n^2}}{s}\right)=H\omega_n\left(\int_0^t\mathcal{L}^{-1}\left(\frac{\omega_n}{s^2+\omega_n^2}\right)(d\tau)\right)$$

$$y(t)=H\omega_n\int_0^t \sin\omega_n\tau d\tau=H\omega_n\frac{1}{\omega_n}(-\cos\omega_n\tau)\Big|_0^t=H(1-\cos\omega_n t)$$



Răspunsul indicial al unui sistem de ordinul 2 fără factor de amortizare (ξ=0)

$$G(s) = \frac{1}{s^2 + 1}$$

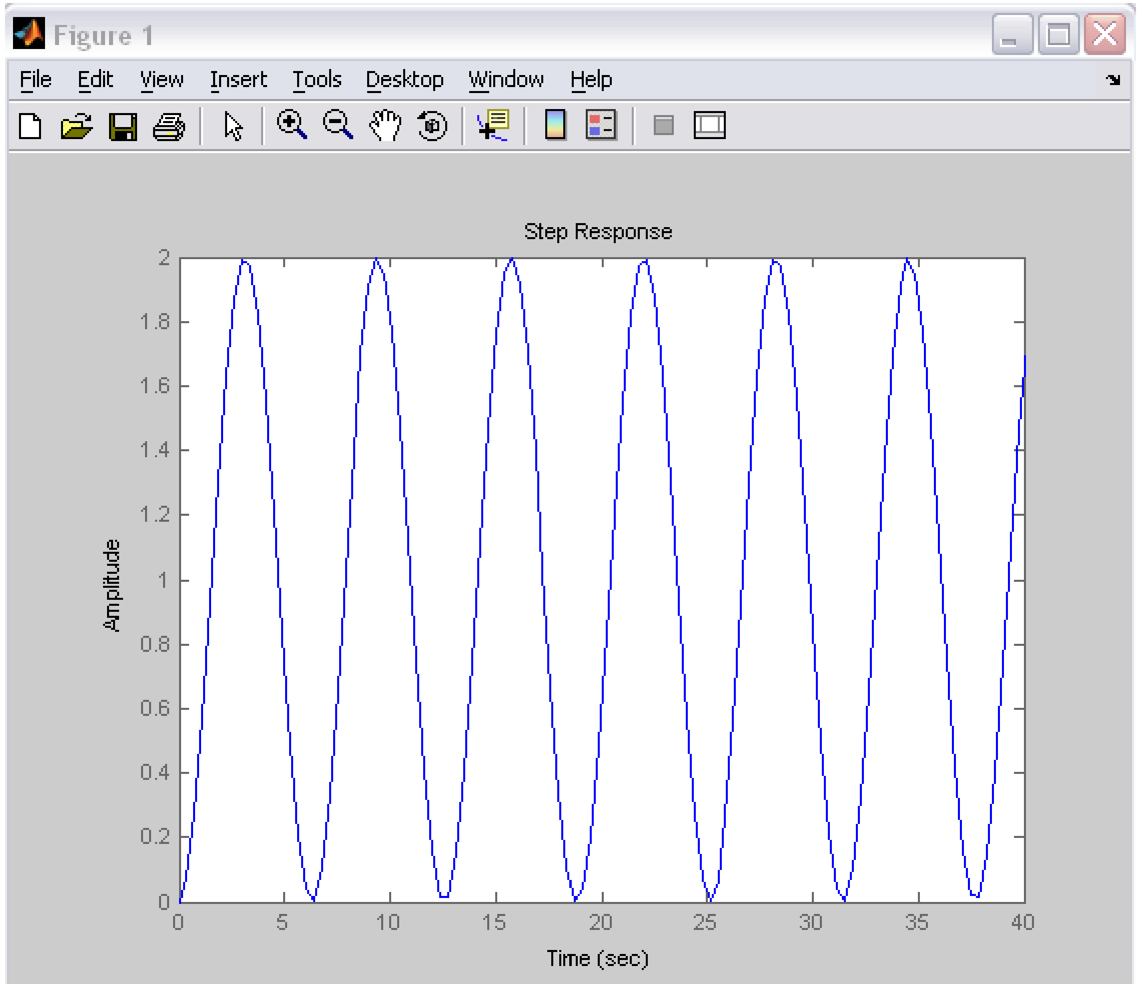
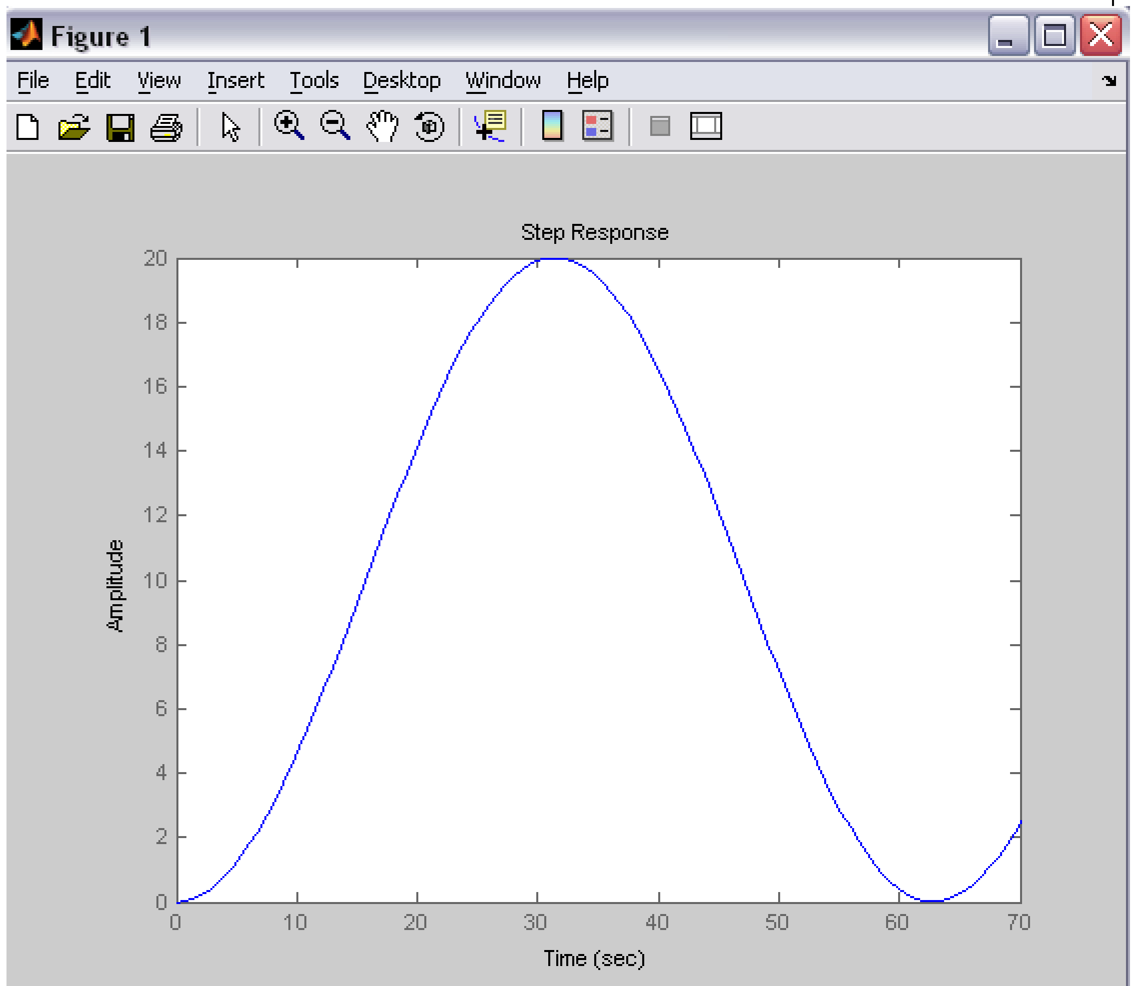


Figure 1
File Edit View Insert Tools Desktop Window Help
Step Response
Amplitude
Time (sec)
0 0.2 0.4 0.6 0.8 1 1.2 1.4 1.6 1.8 2
0 5 10 15 20 25 30 35 40

$$G(s)=\frac{0.1}{s^2+0.01}$$



$$G(s) = \frac{10}{s^2 + 100}$$